International Journal of Theoretical Physics, Vol. 44, No. 8, August 2005 (© 2005)
DOI: 10.1007/s10773-005-4466-7

A Scheme for Physical Implementation of a Ququadrit Quantum Computation with Cooled-Trapped Ions

Chang-yong Chen[1,2,3] and Ke-lin Gao[1]

Received December 4, 2004; accepted March 21, 2005

The physics realization of a ququadrit quantum computation with cooled trapped $^{138}\mathrm{Ba}^+$ ions in a Paul trap is investigated. The ground state level $6^2S_{1/2}(m = -1/2)$ and three metastable levels: $5^2D_{3/2}(m = -1/2)$, $5^2D_{5/2}(m = -1/2)$, and $5^2D_{5/2}(m = 1/2)$, of the fine-structure of the $^{138}\mathrm{Ba}^+$ ion, are used to store the quantum information of ququadrits. The use of coherent manipulation of populations in single ququadrit, being a four-dimensional Hilbert space, produces a discrete Fourier transform and the manipulation of the first red band transitions with the introduction of an ancillary quantum channel between two ququadrits generates a conditional phase gate. The combination of the both above results in a universal two-ququadrit gate, called $XOR^{(4)}$ gate corresponding to the controlled-NOT gate operation in qubit systems. The implementation of quantum Fourier transform for n ququadrits is performed by means of the conditional phase-shift gate. The feasibility of physical realization of ququadrit quantum computation with cooled-trapped $^{138}\mathrm{Ba}^+$ ions is detailed analyzed and described, and the theoretical detection method of logical states is given. Higher entanglement between ququadrits than qutrits or qubits and more security of ququadrit quantum cryptography than qutrit's or qutrit's will lead to more extensive applications ququadrits in quantum information fields. In particular, it is pointed out that this scheme should be the highest dimensional quantum computation in cooled-trapped ions, the entanglement between ququadrits should be the highest dimensional entanglement in it, and the ququadrit quantum cryptography should be the most secure cryptography protocol in it.

KEY WORDS: ququadrit; entanglement; fourier quantum transformation; raman transition.

[1] State Key Laboratory of Magnetic Resonance and Atomic and Molecular Physics, Wuhan Institute of Physics and Mathematics, Center for Cold Atom Physics, The Chinese Academy of Science, Wuhan 430071, P. R. China.
[2] Department of Physics and Electronics and Information and Engineering, Hunan University of Humanity and Science and Technology, Loudi 417000, P. R. China.
[3] To whom correspondence should be addressed at the Department of Physics and Electronics and Information and Engineering, Hunan University of Humanity and Science and Technology, Loudi 417000, P. R. China; e-mail: chenchangyong_64@hotmail.com.

1. INTRODUCTION

The use of cooled-trapped ions for qubit quantum computation was proposed first by Cirac and Zoller (1995). The quantum information is encoded into the long-lived states of the ions and the controlled-NOT gate operation between qubits is accomplished with laser excitation involving the electronic internal states of individually addressed ions and the common quantized vibrational modes of the ion string in the trap. Laser cooled-trapped ions are ideally suited for the investigation and implementation of quantum information processing (Šašura and Bužekaura, 2002) due to localization of a particle less than a few tens of nanometers (Keller *et al.*, 2001) control of the motional state down to the zero point of the trapping potential (Meekhof *et al.*, 1996), a high degree of isolation from environment and thus a very long available time for the manipulation of their quantum state, and the ability to measure the quantum state with high precision by electron shelving technique (Dehmelt, 1975; Monroe *et al.*, 1996). The entanglement in high dimensions is utilized for the realization of new types of quantum communication complexity protocols (Brukner *et al.*, 2002) and provides more security in quantum communication applications (Bechmann-Pasquinucci and Peres, 2000; Bruss and Macchiavello, 2002). The experimental generation of entangled qutrits using two-photon states from the parametric down-conversion process and the experimental realization of entanglement concentration of entangled qutrits of orbital angular momentum entangled photons (Vaziri *et al.*, 2002, 2003) are demonstrated and make the great progress in the use of entanglement in higher dimensions. Recently, the physical implementation of qutrit computer with the trapped ions proposed in Ref. (Klimov *et al.*, 2003) predicts that the physical system of trapped ions can be of extensive applications in the of higher-entanglement quantum communication.

In this paper, the physical implementation of ququadrit quantum computation with laser-cooled-trapped $^{138}Ba^+$ ions in a Paul trap is studied. The ququadrits are defined according to electronic fine-structure levels of trapped $^{138}Ba^+$, composed of the ground level and three metastable levels of it, these levels spanned into a four-dimensional Hilbert space. The accomplishment of quantum gates necessary for a ququadrit quantum computation, including a single ququadrit gate and a conditional quantum gate between two ququdrits, which is called $XOR^{(4)}$, corresponding to the controlled-NOT gate in qubit systems, are performed. Furthermore, a scheme for realization of a quantum Fourier transform for n ququadrits is obtained. In particular, the feasible detection method of the quantum states of ququadrit in the computational basis is proposed in terms of the application of polarization-sensitive laser-induced fluorescence technique (Neuhaustatt *et al.*, 1980; Raab *et al.*, 1998). Higher entanglement between ququadrits than qutrits or qubits and more security of ququadrit quantum cryptography than qutrit's or qutrit's will lead to more extensive applications ququadrits in quantum information fields. it is pointed out that this scheme should be the highest dimensional quantum computation in cooled-trapped ions, the entanglement between ququadrits should be the highest dimensional entanglement in it, and the ququadrit quantum cryptography should be the most secure cryptography protocol in it.

2. THE PHYSICAL SYSTEM AND DETECTION METHOD OF QUANTUM STATES

In this section, the physical basis of the realization of a ququadrit quantum computation, including the physical system of cooled-trapped $^{138}Ba^{+}$ in a Paul trap and the measurement scheme on identifying of the logical states in a ququdrit are considered. A sequence of cooled-trapped $^{138}Ba^{+}$ ions is obtained by the methods and techniques of Ref. (Chu and Wieman, 1989; Nagerl *et al.*, 1999; Resis *et al.*, 1996; Schbert *et al.*, 1995) that have been verified by many well known experiments. Throughout this paper, the logical states or the computational basis of a single ququadrit is represented as $\{|0\rangle, |1\rangle, |2\rangle\}$, where $|0\rangle \equiv 6^2S_{1/2}(m = -1/2)$, $|1\rangle \equiv 5^2D_{3/2}(m = -1/2)$, $|2\rangle \equiv 5^2D_{5/2}(m = -1/2)$, and $|3\rangle \equiv 5^2D_{3/2}(m = 1/2)$. $|0\rangle$ is the ground state while $|1\rangle, |2\rangle, |3\rangle$ are the metastable states of trapped $^{138}Ba^{+}$ ion. The other states in this physical system are defined as $|4\rangle \equiv 6^2P_{1/2}(m = 1/2)$, $|5\rangle \equiv 6^2P_{3/2}(m = -3/2)$, $|6\rangle \equiv 6^2P_{3/2}(m = -1/2)$, $|7\rangle \equiv 6^2P_{1/2}(m = -1/2)$, and $|0'\rangle \equiv 5^2D_{3/2}(m = 1/2)$, respectively, where $|4\rangle$, $|5\rangle$, $|6\rangle$, and $|7\rangle$ are dipole transition excitation states while $|0'\rangle$ is the metastable state. The main fine-structure levels of $^{138}Ba^{+}$ are depicted in Fig. 1 while the corresponding logical states and other related states in a ququadrit are shown as Fig. 2. Then three Raman transition configurations, existing independently among logical states $|0\rangle, |1\rangle$, and $|4\rangle$; $|0\rangle, |2\rangle$, and $|5\rangle$; and $|0\rangle, |3\rangle$, and $|6\rangle$, are driven by (σ^+, σ^+), (σ^-, σ^-), and (π, σ^-) pair-polarized classical

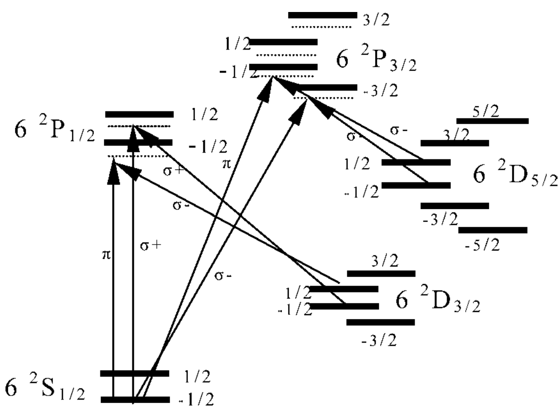

Fig. 1. Raman configurations for defining the logical states of a ququadrit

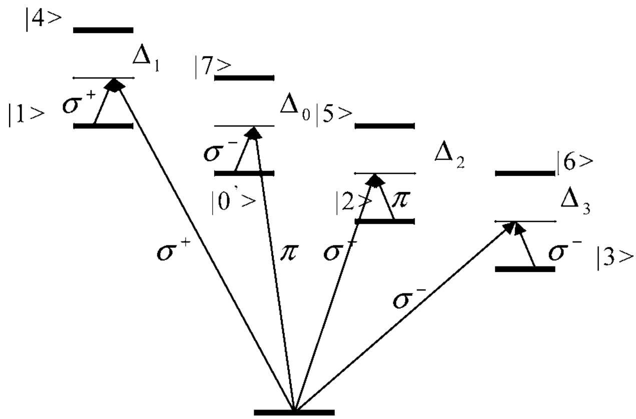

Fig. 2. Electronic fine level structure of trapped ion. Quantum information of ququadrits is stored in levels $|0\rangle$, $|1\rangle$, $|2\rangle$, and $|3\rangle$. Transitions involving effective interactions between levels $|0\rangle \rightarrow |1\rangle$, $|0\rangle \rightarrow |2\rangle$, and $|0\rangle \rightarrow |3\rangle$ are driven by classical fields with different polarizations (σ^+, σ^+), (π, σ^+) and (σ^-, σ^-), respectively.

light fields or electromagnetic fields, respectively. Naturally, if only three Raman transition configurations are used, this resulting in the presence of phases among them, an additional Raman configuration consisting of $|0\rangle$, $|0'\rangle$, and $|7\rangle$ is required for achieving the controlled quantum gate operation of eliminating the phases. This is an ancillary single-ion operation. Diode lasers (Appasamy *et al.*, 1995; Brewer *et al.*, 1992) at 493 and 585 nm introduced by C. Raab *et al.* or dye lasers (DiVincenzo, 2000) at 650, 614, and 445 nm used in the first single-ion experiments with barium ion by W. Neuhauser *et al.*, are available for reaching the interactions mentioned above. Assume that the magnetic fields used for the Zeeman splitting are approximately 10 G, the related energy differences between two neighboring Zeeman sublevels are $E_1 = 0.14|\vec{B}|$, $E_2 = \frac{2}{3} \times 0.14|\vec{B}|$, $E_3 = \frac{4}{5} \times 0.14|\vec{B}|$, $E_5 = \frac{6}{5} \times 0.14|\vec{B}|$, for these levels ${}^2S_{1/2}$, ${}^2P_{1/2}$, ${}^2D_{3/2}$, ${}^2D_{5/2}$, respectively, in which the unit is MHz/G. According to these values, it is clear that the typical energy difference is of few MHz. The trap oscillation frequency suitable for trapping ${}^{138}Ba^+$ ions and the transition frequency between the levels $S \Leftrightarrow P$ and $D \Leftrightarrow P$ reported are of the order of tens of MHz and hundreds of MHz in Klimov *et al.* (2003) and Raab *et al.* (1998), respectively.

In addition, according to the essential criteria (Xu *et al.*, 1996), it is important that the readout of the logical states in computational basis for a universal quantum computation is identified by some effective detection methods. In other words, it is necessary for us to provide a measurement scheme to discriminate $|0\rangle$, $|1\rangle$, $|2\rangle$, and $|3\rangle$. In our case, the transitions: $|0\rangle \rightarrow |1\rangle$, $|0\rangle \rightarrow |2\rangle$, and

$|0\rangle \rightarrow |3\rangle$, are electrically dipole forbidden, which are addressed with Raman transitions in terms of the independent channels related orthogonal polarizations and driven by classical fields, whose channels are (σ^+, σ^+), (σ^-, σ^-), and (π, σ^-), respectively. Our measurement scheme is based on polarization-sensitive laser-induced fluorescence techniques (Oberst, 1999; Xu and Cooke, 1993), which is the development of the electron shelving technique in Dehmelt (1975). When dipole resonant interactions from $|1\rangle \rightarrow |4\rangle$, $|2\rangle \rightarrow |5\rangle$, and $|3\rangle \rightarrow |6\rangle$ are sequentially excited with σ^+-, σ^--, and π-polarized lasers near 650, 614, and 614 nm (Appasamy *et al.*, 1995; Brewer *et al.*, 1992; DiVincenzo, 2000), respectively, the polarized-fluorescence will be sequentially monitored. If the polarizations of emitting fluorescence are sequentially either σ^+ near 493 nm, or σ^- near 455 nm, or π near 455 nm, the logical states are $|1\rangle$, $|2\rangle$, and $|3\rangle$, respectively; Otherwise, that is, non-fluorescence indicates that the logical state is the $|0\rangle$. The detailed and specific detection procedure is described as follows. When every single computation run is over, the Raman transition of the ancillary level system consisting of $|0\rangle$, $|0'\rangle$, and $|7\rangle$ first is driven by (π, σ^+)-polarized Raman channel lasers for some time and then is switched in order to ensure no populations in the ground. Then the transition$|1\rangle \rightarrow |4\rangle$ is continuously driven by σ^+-polarized laser, the Raman transition $|0\rangle \rightarrow |1\rangle$ is continuously driven by $(\sigma^{+,} \sigma^+)$ polarized channel lasers as a re-pumping lasers, and the short-lived $|4\rangle$ will be excited and then scatter photons. Although the efficiency for detecting the photon from one decay of $|4\rangle$ is low, however, one can keep re-pumping the system and scatter millions of photons, eventually detecting a few of them in the case that the state is $|1\rangle$. Otherwise, turning off the manipulation of the system of $|0\rangle$, $|1\rangle$, and $|4\rangle$, the transition $|2\rangle \rightarrow |5\rangle$ is sequentially driven by σ^+-polarized laser, the Raman transition $|0\rangle \rightarrow |2\rangle$ is continuously driven by (π, σ^-)-polarized channel lasers as re-pumping lasers, and the short-lived $|5\rangle$ will be excited and scatter photons. If fluorescence occurs, the state is $|2\rangle$. Or, switching off the manipulation of the system of $|0\rangle$, $|2\rangle$, and $|5\rangle$, the transition $|3\rangle \rightarrow |6\rangle$ is sequentially driven by σ^--polarized laser, the Raman transition $|0\rangle \rightarrow |3\rangle$ is driven by (σ^-,σ^-)-polarized channel lasers as re-pumping lasers, and the short-lived $|6\rangle$ will be excited and scatter photons. If fluorescence is detected, the state is $|3\rangle$. If no fluorescence will occur, the state is $|0\rangle$. In every cycle experiment end the answer will be either $|1\rangle$ firstly when the σ^+-photons are detected or $|2\rangle$ secondly when the π-photons are detected or $|3\rangle$ thirdly when the σ^--photons are detected or $|0\rangle$ finally when no photons are detected, thus distinguishing these logical states with 100% detection efficiency. Actually, our detection scheme is composed of sequential three 'like-electronics shelving' method processes in Dehmelt (1975) with introduction of distinctions of polarization-sensitive lasers. Therefore, the readout of the quantum states is theoretically feasible and the measurement scheme on a ququadrit based on $^{138}Ba^+$ ion is achieved. It is worthwhile pointed out that according to analyzing the level structure of $^{138}Ba^+$ and consideration of only having three orthogonal polarization lasers σ^+, π, and σ^-, it is possible that our computation scheme should be the highest dimensional in scope of trapped-ions. Due to the limitation, it is very difficult for us to conduct more than four-dimensional computation in this scope.

From all of these physical parameters and the measurement scheme mentioned above, it is concluded that the realization of a ququadrit computation using cooled-trapped $^{138}Ba^+$ ions should be feasible in current or planed technology.

3. SINGLE QUQUADRIT GATE

According to Fig. 2, the transitions $|0\rangle \to |1\rangle$, $|0\rangle \to |2\rangle$, and $|0\rangle \to |3\rangle$, being dipole forbidden, are implemented with Raman transitions of independent channels related to orthogonal polarizations, which are driven be classical fields Ω_{04} and Ω_{14}, Ω_{05} and Ω_{15}, Ω_{06} and Ω_{16}, respectively. The ion level populations are adjusted and controlled by selecting the desired coherent operation in this system. The Hamiltonian representing this system, in the rotation wave and dipole approximation, is expressed as

$$H = \sum_{j=0}^{6} \hbar\omega|j\rangle\langle j| + \hbar\left\{e^{-i\nu_2 t}\sum_{i=4}^{6}\Omega_{0i}|i\rangle\langle 0| + e^{-i\nu_1 t}\sum_{i=1}^{6}\Omega_{i+3,i}|i+3\rangle\langle i| + H.C\right\}, \tag{1}$$

In the case of single-ququadrit rotations, only the carrier transition in the ion is considered, so that there is no explicit influence included on the center-of-mass motion of the ion. Thus, the spatial dependences of Raman fields have been included as phase factors. Through adjusting the relationships among the frequencies in order to the detuning being much larger than the Raman frequencies, adiabtically eliminating rapidly decaying upper leves $|4\rangle$, $|5\rangle$, $|6\rangle$, thus we obtain an effective Hamiltonian:

$$\frac{H}{\hbar} = -\sum_{i=1}^{3}\frac{|\Omega_{i+3,i}|^2}{\Delta}|i\rangle\langle i| - \sum_{i=4}^{6}\frac{|\Omega_{i,0}|^2}{\Delta}|0\rangle\langle 0| - \left\{\sum_{i=4}^{6}\frac{\Omega_{i,i-3}\Omega_{i,0}^*}{\Delta}|0\rangle\langle i-3| + H.C\right\}, \tag{2}$$

If the additional condition: $|\Omega_{4,1}|^2/\Delta = |\Omega_{5,2}|^2/\Delta = |\Omega_{6,3}|^2/\Delta = (|\Omega_{4,0}|^2 + |\Omega_{5,0}|^2 + |\Omega_{6,0}|^2)/\Delta$ is satisfied, after some complex calculations, the evolution operator in the four-dimensional space $\{|3\rangle, |2\rangle, |1\rangle, |0\rangle\}$ is expressed as

$$U(\varphi) = \begin{pmatrix} 1+|g|^2c(\varphi) & gg'^*c(\varphi) & gg''^*c(\varphi) & -ig\sin\varphi \\ g^*g'c(\varphi) & 1+|g'|^2c(\varphi) & g'g''^*c(\varphi) & -ig'\sin\varphi \\ g^*g''c(\varphi) & g'^*g''c(\varphi) & 1+|g''|^2c(\varphi) & -ig''\sin\varphi \\ -ig^*c(\varphi) & -ig'^*c(\varphi) & -ig''^*c(\varphi) & \cos\varphi \end{pmatrix}, \tag{3}$$

where $\varphi = \Omega t$, $c(\varphi) = \cos\varphi^{-1}$, and $\Omega^2 = |k''|^2 + |k'|^2 + |k|^2$. The notion $g = k/\Omega$, $g' = k'/\Omega$, and $g'' = k''/\Omega$, where $k = \Omega_{6,0}\Omega^*_{6,3}/\Delta$, $k' = \Omega_{5,0}\Omega^*_{5,2}/\Delta$, and $k'' = \Omega_{4,0}\Omega^*_{4,1}$, is introduced. We can implement all the required coherent operators between any two logical states by the use of this evolution operator. For example, we can activate the transition $|2\rangle \to |3\rangle$, assuming $\varphi = \pi$, $k'' = 0$ in the Eq. (3), then

$$U_1 = \begin{pmatrix} \cos\alpha_1 & -e^{i\beta_1}\sin\alpha_1 & 0 & 0 \\ -e^{-i\beta_1}\sin\alpha_1 & -\cos\alpha_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \tag{4}$$

with defining $\cos\alpha_1 = (|k'|^2 - |k|^2)/(|k'|^2 + |k|^2)$ and $-e^{-i\beta_1} = kk'^*/|kk'^*|$. Other transitions are achieved by adjusting the k, k', k'' constants and φ. The transitions $|1\rangle \to |3\rangle$ with $\varphi = \pi$, $k' = 0$, $|1\rangle \to |2\rangle$ with $\varphi = \pi$, $k = 0$, $|0\rangle \to |1\rangle$ with $k = 0$, $k' = 0$, $|0\rangle \to |2\rangle$with $k = 0$, $k'' = 0$, and $|0\rangle \to |3\rangle$ with $k'' = 0$, $k' = 0$, are addressed, respectively. For the sake of simplicity, Their detail expressions are omitted, represented by U_2, U_3, U_4, U_5, U_6, respectively. Therefore, six operations involving 12 independent parameters are given. In order to generate any SU(4) operator (Vilenkin and Klimyk, 1991), the other three parameters required are obtained by the dispersive evolution U_D connecting interactions of fields with $|0\rangle \to |1\rangle$, $|0\rangle \to |2\rangle$, and $|0\rangle \to |3\rangle$ transitions in the far off-resonance limit. The unitary discrete quantum Fourier transform for a single ququadrit is defined by

$$F|j\rangle \equiv |\bar{j}\rangle = \frac{1}{2}\sum_{l=0}^{3} e^{i2\pi lj/4}|l\rangle, \tag{5}$$

In principle, the Fourier transform is transferred into the form

$$F = iU_D U_6 U_5 U_4 U_3 U_2 U_1, \tag{6}$$

where each of these operations is obtained from the above process if we give each parameters specific values. In fact, any arbitrary one-ququadrit rotation gate is expressed in this way by Eq. (3).

4. CONDITIONAL GATE BETWEEN TWO QUQUADRITS

The conditional gates between two ququadrits are defined as

$$XOR^{(4)}|i\rangle_m|j\rangle_n = |i\rangle_m|i \oplus j\rangle_n, \tag{7}$$

where $i \oplus j$ denotes the addition, modulo 4, m and n denote ququadrit-m and -n, respectively. It will be shown that the this gate is expressed as $XOR^{(4)} = F_n^{-1}P_{mn}F_n$ with F_n being the discrete quantum Fourier transform for one ququadrit and the P_{mn} a conditional phase-shift gate between ququdrit-m and ququadrit-n. F_n is originated from the coherent manipulation of populations a four-dimensional Hilbert space spanned by the four logical states in a ququadrit being the ground state and the three metastable state of the $^{138}Ba^+$ ion while P_{mn} is produced by the first sideband resonant transitions and the intervention of an ancillary quantum channel between ququadrits.

A fundamental requirement for which we perform a conditional phase gate between two ququadrits is that a mechanism is provided to distinguish independent quantum paths and to satisfy the conditional change in the target ququadrit which depends on the control ququadrit. The definition of the $XOR^{(4)}$ gate is that the target changes only when the control ququadrit is in state $|1\rangle$, $|2\rangle$, and $|3\rangle$, otherwise, the state of the target will be unchanged. Therefore, only a protocol is required to perform in which independent quantum channels through $|1\rangle$, $|2\rangle$, and $|3\rangle$ of the control ququadrit are considered. Such channels can be established with the comprising of the collective center-of-mass (CM) motion of ^{138}Ba ions in the Paul trap.

Assuming that $\Omega_{0,6} = \Omega_{3,6}$, $\Omega_{0,5} = \Omega_{2,5}$ and $\Omega_{0,4}$, $\Omega_{1,4} \neq 0$, or $\Omega_{0,6} = \Omega_{3,6} = 0$, $\Omega_{1,4} = \Omega_{0,4} = 0$, and $\Omega_{0,5}$, $\Omega_{2,5} \neq 0$, and $\Omega_{0,5} = \Omega_{2,5} = 0$, or $\Omega_{0,6} = \Omega_{3,6} = 0$, $\Omega_{1,4} = \Omega_{0,4} = 0$, and $\Omega_{0,6} = \Omega_{3,6} \neq 0$, in three cases, after eliminating the upper excited level and adjusting to the first red sideband transition, The Hamiltonian that describes the ion center of mass coupled to the electronic transition $|0\rangle \rightarrow |q\rangle$ is:

$$H_{n,q} = \frac{\Omega_q \eta}{2} \left[|q\rangle_n \langle 0| a e^{-i\delta t - i\phi} + a^+ |0\rangle_n \langle q| e^{i\delta t + i\phi} \right], \tag{8}$$

Here $q = 1, 2, 3$, represents polarizations. a^+ and a^- are the creation and annihilation operators of the CM phonons, respectively, Ω_q denotes the effective Rabi frequency, ϕ is the laser relative phase, and $\eta = \sqrt{\hbar k_\theta^2/(2M\nu_\theta)}$ is the Lamb-Dicke parameter. We can coherently manipulate the center-of-mass motion coupled to electronic transitions, through the selection of the effective interaction time and laser polarizations. Then, the implementation of the coherent interaction between ququadrits and the collective center-of-mass motion is allowed in terms of this Hamiltonian, whose evolution operation is shown as

$$\begin{aligned} lU_m^{l,q}(\phi)|0\rangle_m\langle 0| &= |0\rangle_m\langle 0|, \\ U_m^{l,q}(\phi)|0\rangle_m\langle 1| &= \cos(\Omega_q \eta t/2)|0\rangle_m\langle 1| - ie^{-i\phi}\sin(\Omega_q \eta t/2)|q\rangle_m\langle 0|, \\ U_m^{l,q}(\phi)|q\rangle_m\langle 0| &= \cos(\Omega_q \eta t/2)|q\rangle_m\langle 0| - ie^{-i\phi}\sin(\Omega_q \eta t/2)|0\rangle_m\langle 1|, \end{aligned} \tag{9}$$

where we assume $\Omega_q \eta t/2 = l\pi/2$ for simplicity. It is clear that the choice of the quantum channels for transferring the information to the center-of-mass is allowed by the use of these coherent operations. Thus, a phase change in ququadrit depending on the energy of the center-of-mass state is necessarily introduced. This phase change is accomplished through the dispersive regime of the first red sideband in Eq. (9), that is,

$$D_m^q(\varphi) = e^{i\varphi aa^+}|q\rangle_m\langle q| + e^{-i\varphi aa^+}|0\rangle_m\langle 0|, \tag{10}$$

where $\varphi = (\Omega_q\eta)^2/4\delta$, allowing for an intensity-dependent phase shift of the fine-structure levels.

From Eq. (9), the conditional phase shift needed to realize the $XOR^{(4)}$, in the computational basis $\{|0\rangle_m|0\rangle_n, |0\rangle_m|1\rangle_n, \ldots, |3\rangle_m|3\rangle_n\}$ is expressed as:

$$P_{m,n} = P_{mn}^{(3)}P_{mn}^{(2)}P_{mn}^{(1)} = \text{dig}(1, 1, 1, 1, 1, i, -1, -i, 1, -1, 1, -1, 1, -i, -1, i), \tag{11}$$

where

$$\begin{aligned} P_{mn}^{(1)}(\phi_1, \phi_2, \phi_3) = \ & R_{00'}(\pi)U_m^{1,1}(3\pi/2)D_n^{\prime 3}(\xi_3)D_n^3(\phi_3)D_n^{\prime 2}(\xi_2)\, D_n^2(\phi_2)D_n^{\prime 1}(\xi_1) \\ & \times D_n^1(\phi_1)U_m^{1,1}(\pi/2)R_{00'}(\pi), \\ P_{mn}^{(2)}(\phi_2, \phi_3, \phi_1) = \ & R_{00'}(\pi)U_m^{1,2}(3\pi/2)D_n^{\prime 3}(\xi_1)D_n^3(\phi_1)D_n^{\prime 2}(\xi_3)D_n^2(\phi_3)D_n^{\prime 1}(\xi_2) \\ & \times D_n^1(\phi_2)U_m^{1,2}(\pi/2)R_{00'}(\pi), \\ P_{mn}^{(3)}(\phi_3, \phi_2, \phi_1) = \ & R_{00'}(\pi)U_m^{1,3}(3\pi/2)D_n^{\prime 3}(\xi_2)D_n^3(\phi_2)D_n^{\prime 2}(\xi_1)D_n^2(\phi_1)D_n^{\prime 1}(\xi_3) \\ & \times D_n^1(\phi_3)U_m^{1,3}(\pi/2)R_{00'}(\pi), \end{aligned} \tag{12}$$

with $\xi_i = 2\pi - \phi_i$. The operation $R_{00'}(\pi)$ is a rotation only impinging on the ion when it is in the $|0\rangle$ level, which sends it to $|0'\rangle$ level preventing from the generation of any phase shift in this state. The expressions of the dispersive operations influencing on transitions $|0\rangle \rightarrow |1\rangle$, $|0\rangle \rightarrow |2\rangle$, $|0\rangle \rightarrow |3\rangle$ are $D_n^1(\phi_1)$, $D_n^{\prime 1}(\varphi_1)$, $D_n^2(\phi_2)$, $D_n^{\prime 2}(\varphi_2)$, $D_n^3(\phi_3)$, $D_n^{\prime 3}(\varphi_3)$ that are orthogonal matrices. Their detail formulae are omitted for simplicity (see Oberst, 1999).

Thus, the particular phase-shift gate in Eq. (11) can be reached by $P_{mn}^{(1)}(\pi/2, \pi, 3\pi/2)$, $P_{mn}^{(2)}(\pi, 2\pi, 3\pi)$, and $P_{mn}^{(3)}(\pi, 0, \pi)$. Finally, the combination of the effective conditional change of the target ququadrit state with the discrete quantum Fourier transform constructs the $XOR^{(4)}$ gate which, in the computational basis $\{|0\rangle_m|0\rangle_n, |0\rangle_m|1\rangle_n, \cdots, |3\rangle_m|3\rangle_n\}$, is specifically expressed as

$$XOR^{(4)} = F_n^{-1} P_{mn}^{(3)} P_{mn}^{(2)} P_{mn}^{(1)} F_n = \begin{bmatrix} 1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0 \\ 0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0 \\ 0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1 \\ 0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0 \end{bmatrix}, \tag{13}$$

5. FOURIER TRANSFORM FOR MANY-QUQUADRITS

In this section, as the first application, the general protocol for the quantum Fourier transform for a quantum system of n ququadrits is proposed. The Fourier transform in four-dimensional Hilbert space (Nielsen and Chuang, 2000) is expressed as:

$$|\bar{j}\rangle = \frac{1}{4^{n/2}} \sum_{k=0}^{4^n-1} e^{i2\pi j(k/4^n)} |k\rangle, \tag{14}$$

where $0 \leq j \leq 4^n - 1$. An equivalent product form is shown as

$$\begin{aligned} |\bar{j}\rangle = \; & \frac{1}{4^{n/2}} \left[\sum_{k_1=0}^{3} e^{(i2\pi j k_1 O j_n)} |k\rangle \right] \left[\sum_{k_2=0}^{3} e^{(i2\pi j k_2 O j_{n-1} j_n)} |k_2\rangle \right] \cdots \\ & \times \left[\sum_{k_n=0}^{3} e^{(i2\pi j k_n O j_1 j_2 \ldots j_n)} |k_n\rangle \right], \end{aligned} \tag{15}$$

If the summation for each factor is expended, for instance, the last term is written as follows:

$$\sum_{k_n=0}^{3} e^{i2\pi j k_n O j_1 j_2 \ldots j_n}|k_n\rangle = \frac{1}{2}\big[|0\rangle + e^{i2\pi O j_1 j_2 \ldots j_n}|1\rangle + e^{i4\pi O j_1 j_2 \ldots j_n}|2\rangle$$
$$+ e^{i6\pi O j_1 j_2 \ldots j_n}|3\rangle\big] = \frac{1}{2}\big[|0\rangle + e^{i2\pi(j_1/4 + j_2/4^2 + \cdots + j_n/4^n)}|1\rangle$$
$$+ e^{i4\pi(j_1/4+\cdots+j_n/4^n)}|2\rangle + e^{i6\pi(j_1/4+\cdots+j_n/4^n)}|3\rangle\big], \qquad (16)$$

This state can be created by the starting of the application of a Fourier transform on the first ququadrit

$$F_1^{(4)}|j_1\rangle|j_2\rangle \ldots |j_n\rangle = \frac{1}{2}\big[|0\rangle + e^{i2\pi j_1/4}|1\rangle$$
$$+ e^{i4\pi j_1/4}|2\rangle + e^{i6\pi j_1/4}|3\rangle\big]|j_2\rangle|j_3\rangle \ldots |j_n\rangle, \qquad (17)$$

and then the application of conditional phase transformations on this ququadrit state, which we turn to the initial state of remaining $|j_2\rangle, |j_3\rangle, \ldots, |j_n\rangle$ ququadrit states,

$$R_{n1} \ldots R_{31} R_{21} F_1^{(4)}|j_1\rangle|j_2\rangle \ldots |j_n\rangle = \frac{1}{2}\big[|0\rangle + e^{i2\pi(j_1/4+\cdots+j_n/4^n)}|1\rangle$$
$$+ e^{i4\pi(j_1/4+\cdots+j_n/4^n)}|2\rangle$$
$$+ e^{i6\pi(j_1/4+\cdots+j_n/4^n)}|3\rangle\big]|j_2\rangle|j_3\rangle \ldots |j_n\rangle, \qquad (18)$$

The conditional phase is as follows

$$R_{21} = P_{21}^{(3)}\left(\frac{6\pi}{4^2}, \frac{12\pi}{4^2}, \frac{18\pi}{4^2}\right) P_{21}^{(2)}\left(\frac{4\pi}{4^2}, \frac{8\pi}{4^2}, \frac{12\pi}{4^2}\right) P_{21}^{(1)}\left(\frac{2\pi}{4^2}, \frac{4\pi}{4^2}, \frac{6\pi}{4^2}\right),$$
$$R_{31} = P_{31}^{(3)}\left(\frac{6\pi}{4^2}, \frac{12\pi}{4^2}, \frac{18\pi}{4^2}\right) P_{31}^{(2)}\left(\frac{4\pi}{4^2}, \frac{8\pi}{4^2}, \frac{12\pi}{4^2}\right) P_{31}^{(1)}\left(\frac{2\pi}{4^2}, \frac{4\pi}{4^2}, \frac{6\pi}{4^2}\right),$$
$$\cdots$$
$$R_{k1} = P_{k1}^{(3)} P_{k1}^{(2)} P_{k1}^{(1)},\ P_{k1}^{(j_k)} = P_{k1}^{(j_k)}\left(\frac{2 j_k \pi}{4^{j_k}}, \frac{4 j_k \pi}{4^{j_k}}, \frac{6 j_k \pi}{4^{j_k}}\right), \qquad (19)$$

In the same methods, the state

$$\sum_{k_n=0}^{3} e^{i2\pi j k_n O j_1 j_2 \ldots j_n}|k_n\rangle = \frac{1}{2}\big[|0\rangle + e^{i2\pi(j_1/4 + j_2/4^2 + \cdots + j_n/4^n)}|1\rangle$$
$$+ e^{i4\pi(j_1/4+\cdots+j_n/4^n)}|2\rangle + e^{i6\pi(j_1/4+\cdots+j_n/4^n)}|3\rangle\big], \qquad (20)$$

is produced by the application of a Fourier transform on the ququadrit $|j_2\rangle$ and then conditional phase operation on this ququadrit state are performed, which we turn to the state of the remaining $|j_3\rangle, \ldots, |j_n\rangle$ ququadrit states. Therefore, the n-ququadrit quantum Fourier transformation is reached at the end.

6. SOME POSSIBLE APPLICATIONS OF QUQUADRITS

Some possible applications of ququadrats in information field, such as the security of quantum key distribution, the effective resources, and the separable and the nonseparable conditions of entanglement among ququadrits are shown in this section. According to Bruss and Macchiavello (2002) and Bechmann-Pasquinucci and Peres (2000), theoretically, the use of higher-dimensional quantum systems instead of lower-dimensional systems can be improve the security of quantum key distribution and increase the ability against the individual attacks, the coherent attacks, and the symmetric attacks. Here we provide a specific real physical system, ququadrat, consisting of trapped $^{138}Ba^{+}$ ions. Therefore, it is predicted that the corresponding experimental demonstration on the security of key distribution with ququadrit systems should be implemented according to our scheme in the context of trapped ions. The security can improve approximately 20% comparable to that of qutrits or qubits. Furthermore, the implementation of quantum computation complexity protocol of two-entangled-ququdrats is more efficient than that of qutrits or qubits, derived from Brukner *et al.* (2002). In the other words, there exist more available resources in a ququadrit than in a qutrit or qubit, which, in a d-dimensional quantum system, are defined as $R = \ln d$, meaning R increases along with d. According to Carlton and Miburn (2000) and Nicolas *et al.* (2002), the fact that the two-qutrit mixture is separable if and only if the possibility ε_3 for the maximally entanglement does not exceed 1/4, indicates that maximally entangled states of qutrits are more entangled than the maximally entangled states of two qubits, because $\varepsilon_2 \leq 1/3$. In fact, in the similar situation, the maximally entangled states between ququadrits are more entangled than those between qutrits or qubits, because $\varepsilon_4 \leq 1/5$. At the same time, the separable condition $\varepsilon_4 \leq 1/(1+4^{2n-1})$ can be obtained by the extension of the methods in Carlton and Miburn (2000) and Nicolas *et al.* (2002), when n-ququdrits are a mixture of a maximally mixed state and maximally entangled state $\rho_{\varepsilon_4} = (1-\varepsilon_4)M_{d^2} + \varepsilon_4\rho_1$ where $M_{4^n} = 1 \otimes 1/4 \otimes \cdots \otimes 1/4^n$ the maximally mixed for n ququdrits, and ρ_1 is any n-ququadrit density matrix. It is clear that $\varepsilon_{4\,\max} < \varepsilon_{3\,\max} < 3_{2\,\max}$, where $\varepsilon_3 \leq 1/(1+3^{2n-1})$, $\varepsilon_2 \leq 1/(1+2^{2n-1})$. Whereas, the non-separable condition $\varepsilon_4 1/(1+4^{n+2})$ can be obtained when the joint density operators $\rho_{\varepsilon_4} = (1-\varepsilon_4)M_{d^2} + \varepsilon_4|\phi\rangle\langle\phi|$ for two ququadrits, where $M_{d^2} = 1/d^2$ is the maximally mixed state in $d^2 = 4^n$ dimension and $|\phi\rangle = (1/\sqrt{d})\sum_{i=1}^{d}|i\rangle\langle i|$ is a maximally entangled state for two particles, is considered. It is apparent that $\varepsilon_{4\,\min} < \varepsilon_{3\,\min} < \varepsilon_{2\,\min}$, in which $\varepsilon_3 \geq 1/(1+3^{n/2})$ and $\varepsilon_2 \geq 1/(1+2^{n/2})$. The combination of the both mentioned above results naturely in a conclusion that the entanglement from ququadrit is more stronger than those from qutrits or qubits. The higher entanglement between ququadrits than qutrits or qubits and more security of ququadrit quantum cryptography than qutrit's or qutrit's will lead to more extensive applications ququadrits in quantum information fields. Owing to the highest-dimension used for quantum computation in the cooled-trapped ions being four, the entanglement between ququadrits is the most entangled and the security of ququadrit quantum cryptography is the most secure in the scope of ions.

7. CONCLUDING REMARKS

In summary, a scheme for the physical implementation of a universal ququadrit quantum computation, whose logical states are the electronic fine-structure levels of cooled-trapped $^{138}Ba^{+}$ ions, which is composed of the single ququadrit gate included also in a discrete quantum Fourier transform, the conditional gate (i.e., $XOR^{(4)}$) between two ququadrits, and the quantum logical state detection methods in the four-dimensional computational basis, has been proposed. The Fourier transform for many-ququadrits is achieved by means of a sequence of conditional phase transformational operations. Although our scheme is the direct extension of that of Klimov *et al.* (2003) from the ways, the extended process is not trial as the same as Klimov *et al.* (2003) is the direct extension of that of Cirac and Zoller (1995). In our paper, the polarized-sensitive stimulated Raman transition techniques (Appasamy *et al.,* 1995; Brewer *et al.*, 1992) and the polarization-laser-induced resonance fluorescence detection techniques (Oberst, 1999; Xu and Cooke, 1993), making in principle our scheme feasible, are utilized for the coherent manipulation of the populations of the logical states of each ququadrit and the discrimination and identifying of the logical states at the end of each computational run, respectively. It is more important that the detailed and specific detection process of them has been given in Section 2. It has been demonstrated that our scheme is the highest dimensional quantum computation in the physical system of cooled-trapped ions duo to the combining consideration of the level structure of $^{138}Ba^{+}$ and three polarization states of lasers so far. It has been shown through calculations that the ququadrit-quantum entanglement with cooled-trapped $^{138}Ba^{+}$ should be more entangled than the qubit- and qutrit-entanglement and be the most dimensional entanglement in cooled-trapped ions, and ququadrit-quantum secure cryptograph should be more security than the qubit- and qutrit-cryptograph protocols, and be the most secure quantum cryptography in cooled-trapped ions (Bechmann-Pasquinucci and Peres, 2000; Bruss and Macchiavello, 2002; Carlton and Miburn, 2000; Nicolas *et al.*, 2002). They can be generated by using the basic methods here and will be found more extensive applications in higher-dimensional quantum communication fields. Indeed, our scheme is easily extended to the situation of cooled-trapped $^{43}Ca^{+}$ ions (Schmidt-Kaler *et al.*, 2003; Steane, 1997).

Some further studies on the applications of the ququadrit quantum computation are beyond this paper, which will be given elsewhere.

ACKNOWLEDGMENTS

We thank Qin Tao, Zhang Xiao-long, Nie Zhong-xiu, and Wei Da-xiu for enlightening discussions. This project is supported in part by the Foundation for Youth Core-Teachers of Hunan Province under Grant No: 2003165, the Science Research Fund of Educational Bureau of Hunan Province of China under Grant No: 03C046, the Science Research Fund of Hunan University of Humanity and Science and Technology under Grant No: 2004A014, The National Science Foundation of China under Grant No: 0274093, and the National Fundamental Research Program under Grant No: 2001CB309309.

REFERENCES

Appasamy, B., Siemers, I., Stalgies, Y., Eschner, J., Blatt, R., Neuhauser, W., and Toschek, P. E. (1995). *Applied Physics B* **60**, 473.
Bechmann-Pasquinucci, H. and Peres, A. (2000). *Physical Review Letters* **86**, 3318.
Brewer, R. G., Devoe, R. G., and Kallenbach, R. (1992). *Physical Review A* **46**, R6781.
Brukner, C., Zukowski, M., and Zeilinger, A. (2002). *Physical Review Letters* **89**, 197901.
Bruss, D. and Macchiavello, C. (2002). *Physical Review Letters* **88**, 127901.
Carlton, M. C. and Miburn, G. J. (2000). *Optics Communication* **179**, 439.
Chu, S. and Wieman, C. (1989). *Journal of the Optical Society of America B* **6**,(11) (special issue on laser cooling and trapping of atoms.).
Cirac, J. I. and Zoller, P. (1995). *Physical Review Letters* **74**, 4091.
Dehmelt, H. (1975). *Bulletin of the American Physical Society* **20**, 60.
DiVincenzo, D. P. (2000). *Fortschritte der Physik* **48**, 771.
Keller, G. R., Hayasaka, M., Lange, K. W., and Walther, H. (2001). *Nature* **414**, 49.
Klimov, A. B., Guzman, R., Retamal, J. C., and Saavedra, C. (2003). *Physical Review A* **67**, 062313.
Meekhof, D. M., Monroe, C., King, B. E., Itano, W. M., and Wineland, D. J. (1996). *Physical Review Letters* **76**, 1796.
Monroe, C., Meekhof, D. M., King, B. E., and Windland, D. J. (1996). *Science* **272**, 1131.
Nagerl, H. C., Leibried, D., Rohde, H., Thalhammer, G., Eschner, J., Schmidt-Kaler, F., and Blatt, R. (1999). *Physical Review A* **60**, 145.
Neuhaustatt, W., Toschek, P. E., and Dehmelt, H. G. (1980). *Physical Review A* **22**, 1137.
Nicolas, J., Bourennae, M., Karlson, A., and Gisin, N. (2002). *Physical Review Letters* **88**, 127902.
Nielsen, M. A. and Chuang, I. L. (2000). *Quantum Computation and Quantum Information*, Cambridge University Press, Cambridge.
Oberst, H. (1999). *Diplomarbeit*, Innsbruck University, Austria.
Raab, C., Bolle, J., Eschner, J., Schmidt-Kaler, F., and Blatt, R. (1998). *Applied Physics B* **67**, 683.
Resis, D., Lindner, A., and Blatt, R. (1996). *Physical Review A* **54**, 5133.
Šašura, M. and Bužek, V. (2002) *Journal of Modern Optics* **49**, 1593.
Schbert, M., Siemers, I., Blatt, R., and Toschek, P. E. (1995). *Physical Review A* **52**, 2994.
Schmidt-Kaler, F., Gulde, S., Riebe, M., *et al.* (2003). *Journal of Physics D* **36**, 623.
Steane, A. (1997) *Applied Physics B* **64**, 623.

Vaziri, A., Pan, J. W., Jennewein, T., Weihs, G., and Zeilinger, A. (2003). *Physical Review Letters* **91**, 227902.

Vaziri, A., Weihs, G., and Zeilinger, A. (2002). *Physical Review Letters* **89**, 240401.

Vilenkin, N. and Klimyk, A. (1991). *Representation of Lie Group and Special Functions*, Vol. 1–3, Kluwer, Dordrecht.

Xu, G. T. and Cooke, W. E. (1993). *Physical Review A* **48**, 2054.

Xu, G. T., Wang, X., and Cooke, W. E. (1996). *Journal of Physics B* **26**, L103.